Journal of Global Optimization **24:** 89–107, 2002.

Cone Adaptation Strategies for a Finite and Exact Cutting Plane Algorithm for Concave Minimization

MARCUS POREMBSKI
University of Marburg, Department of Economics and Business Administration, 35032 Marburg, Germany (e-mail: porembsk@mailer.uni-marburg.de)

Abstract. In this paper we are concerned with pure cutting plane algorithms for concave minimization. One of the most common types of cutting planes for performing the cutting operation in such algorithm is the concavity cut. However, it is still unknown whether the finite convergence of a cutting plane algorithm can be enforced by the concavity cut itself or not. Furthermore, computational experiments have shown that concavity cuts tend to become shallower with increasing iteration. To overcome these problems we recently proposed a procedure, called cone adaptation, which deepens concavity cuts in such a way that the resulting cuts have at least a certain depth Δ with $\Delta > 0$, where Δ is independent of the respective iteration, which enforces the finite convergence of the cutting plane algorithm. However, a crucial element of our proof that these cuts have a depth of at least Δ was that we had to confine ourselves to ε-global optimal solutions, where ε is a prescribed strictly positive constant. In this paper we examine possible ways to ensure the finite convergence of a pure cutting plane algorithm for the case where $\varepsilon = 0$.

Key words: Concave minimization, Cutting plane, Concavity cut, Tuy cut, Finite convergence, Cone adaptation.

1. Introduction

In this paper we are concerned with concave minimization problems of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{s.t.} & x \in P, \end{array} \tag{1.1}$$

where $f : I\!R^n \mapsto I\!R$ is concave on $I\!R^n$ and $P = \{x \in I\!R^n \mid Ax \leq b\}$ is a polyhedron. For the sake of simplicity we assume that P is bounded with $\dim(P) = n$, i.e. P is a full-dimensional polytope, and that $f(x)$ is finite on $I\!R^n$ and not constant on P. Furthermore, we assume that the level sets

$$L(\gamma) = \{x \in I\!R^n \mid f(x) \geq \gamma\} \tag{1.2}$$

are closed and bounded for all real numbers γ. Note that since $f(x)$ is concave, the level sets $L(\gamma)$ are convex.

The methods for solving concave minimization problems fall mainly into three categories: enumerative methods, successive partition methods, and successive approximation methods. For an overview of the different algorithms the reader is referred to Benson (1995, 1996), Horst and Tuy (1996), and Tuy (1998). The successive partition methods and their branch and bound variants are probably the most popular. In this paper we are concerned with cutting plane algorithms, which belong to the class of enumerative methods.

A cutting plane algorithm for concave minimization consists of two alternating phases: ‘search’ and ‘cut’. In the search phase we find a local optimum and in the cut phase we eliminate this local optimum with a cutting plane without excluding a solution with a smaller objective value than the incumbent solution. The algorithm terminates when the feasible region is empty, i.e. when P has been completely eliminated by cutting planes. The incumbent solution is then a global optimum.

One of the most important types of cutting planes in concave minimization is the concavity cut, also known as a convexity cut and a Tuy cut. In general, we use this type of cut to eliminate a local optimum in the cut phase. However, experiments in the mid-1970s (e.g., Zwart, 1971) showed that the concavity cuts tend to become more and more shallow, thereby slowing down the search process. Furthermore, it is still unknown whether a pure cutting plane algorithm that uses only concavity cuts is finitely convergent.

Clearly, a cutting plane algorithm is finitely convergent if there exists a strictly positive constant Δ such that the depth of all cuts is at least Δ (c.f. Horst and Tuy, 1996, Theorem V.2). Based on this observation a procedure was proposed, called cone adaptation (c.f. Porembski, 2001), that deepens concavity cuts in such a way that the resulting cuts have at least this depth. Hence a cutting plane algorithm based on these cuts is finitely convergent.

However, to prove that the cuts derived by cone adaptation have a depth of at least Δ, it is crucial to assume that we confine ourselves to finding an ε-global optimal solution of the concave minimization problem (1.1), i.e., a solution $\widehat{x} \in P$ with $f(\widehat{x}) \leq f(x) + \varepsilon$ for all $x \in P$, where $\varepsilon > 0$ is a prescribed tolerance. Even though in most applications it suffices to search for an ε-global optimal solution, it is also interesting to look for ways to ensure the finite convergence of a cutting plane algorithm for the case where $\varepsilon = 0$.

For concave minimization algorithms one can often find in the literature statements such as: ‘The algorithm is finitely convergent for $\varepsilon > 0$. If $\varepsilon = 0$ the algorithm either terminates at an exact global optimum after a finite number of iterations, or else it involves an infinite sequence that converges to an exact global optimum.’ In recent years there has been some interest in also ensuring finite convergence for $\varepsilon = 0$. For instance, Shectman and Sahinidis (1998) proposed a rectangular algorithm, and Locatelli and Thoai (2000) a simplicial branch and bound algorithm, both of which are finite and exact.

Another interesting approach is that of Al-Khayyal and Sherali (2000): choose a sufficiently small ε with $\varepsilon > 0$, identify an ε-global optimal solution with one of the already known finitely convergent algorithms and, using the ε-global optimal solution as a starting point, in a purification step determine a vertex of P that is an exact global optimum. However, as we have seen in experiments with pure cutting plane algorithms using concavity cuts, when ε decreases the number of cuts needed to solve the concave minimization problem might dramatically increase (c.f. Porembski, 1996), and in the Al-Khayyal and Shearli approach ε is in most cases quite small.

In this paper we propose another modification of the cutting plane algorithm outlined above, to make it finite *and* exact. In addition to their use in pure cutting plane algorithms, concavity cuts have found application in many other kinds of algorithms, for instance conical algorithms and branch and bound algorithms. The ideas presented in this paper might also lead to modifications of these algorithms to make them both finite and exact.

This paper is organized as follows. In the next section we describe the basic idea behind cone adaptation. In the third section we discuss modifications of the cutting plane scheme that ensure finite convergence for the case where $\varepsilon = 0$. In the fourth section we discuss how these modifications can be implemented.

2. Valid cuts and cone adaptation

As already outlined in the Introduction, a cutting plane algorithm for concave minimization consists of the two alternating phases 'search' and 'cut'. In the following we discuss these phases in more detail.

In the search phase we are looking for a local optimum. Here we make use of a well-known feature of concave minimization problems: Since the objective function $f(x)$ is, by assumption, not constant on P, all global minima of problem (1.1) are attained at the boundary of P (c.f., e.g., Mangasarian, 1969, Theorem 5.2.3). Furthermore, there exists a vertex of P which is a global optimum. If $f(x)$ is strictly concave, then the global optimum is always attained at a vertex of P.

Hence we can restrict our search to the vertices $\text{vert}(P_k)$ of P_k, where P_k denotes the polytope we have obtained from P by deriving $k-1$ cutting planes, i.e., we are now in the kth iteration of the cutting plane algorithm. In this context a vertex of P_k is called a *local optimum* or a *star optimum* if its adjacent vertices have no smaller objective value. Hence we can identify a local optimum in the search phase, starting at an arbitrary vertex of P_k, by pivoting from one vertex to the adjacent vertex with the smallest objective value, as long as the objective value strictly decreases. This procedure terminates at a local optimum x_{0_k} after a finite number of iterations. If $f(x_{0_k}) < \widehat{f}$, where $\widehat{f}$ denotes the objective value of the incumbent solution, then we update $\widehat{f}$, i.e., $\widehat{f} := f(x_{0_k})$. Hence we have $x_{0_k} \in L(\widehat{f})$ (see (1.2)).

In the cut phase we eliminate x_{0_k} with a valid cut, i.e., with a cutting plane that eliminates x_{0_k} but no $x \in P_k \setminus L(\widehat{\gamma})$, where $\widehat{\gamma} := \widehat{f} - \varepsilon$ if we content ourselves with an ε-global optimal solution and $\widehat{\gamma} := \widehat{f}$ if we search for an exact global optimum. A valid cut based on cone adaptation is derived in two main steps: In the first a concavity cut is derived and in the second the cut is deepened by application of cone adaptation.

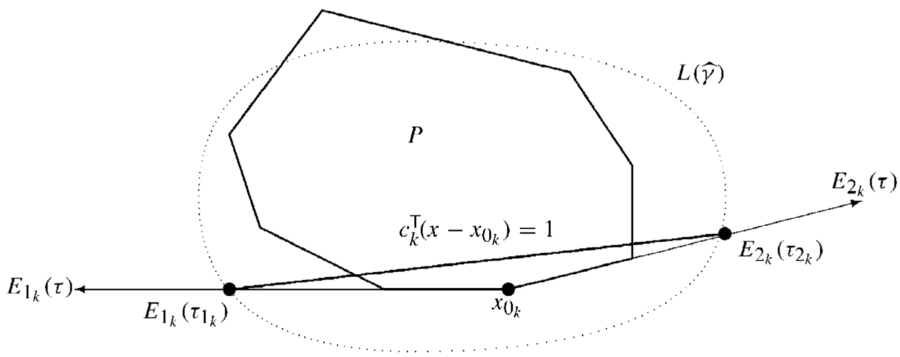

Figure 1. Deriving a concavity cut.

A concavity cut is derived as follows. We consider the P_k-containing cone

$$C(x_{0_k}) = x_{0_k} + \text{cone}(u_{1_k}, u_{2_k}, \dots, u_{r_k}), \tag{2.3}$$

where $u_{1_k}, u_{2_k}, \dots, u_{r_k} \in I\!R^n$ with $r_k \geq n$ are the directions of the edges of P_k emanating from x_{0_k}, and determine the intersection points $E_{i_k}(\tau_{i_k})$ of its edges $E_{i_k}(\tau) = x_{0_k} + \tau u_{i_k}$, $\tau \geq 0$, with the boundary $\text{bd}(L(\widehat{\gamma}))$ of $L(\widehat{\gamma})$. Then we determine a hyperplane $c_k^{\mathsf{T}}(x - x_{0_k}) = 1$ that intersects the edges of $C(x_{0_k})$ in $L(\widehat{\gamma})$ and contains at least n of these points. That is, we determine a basic feasible solution of the system

$$c^{\mathsf{T}}(E_i(\tau_i) - x_{0_k}) \geq 1 \qquad \text{for } i = 1, 2, \dots, r_k \tag{2.4}$$

(c.f. Benson, 1999). Note that if x_{0_k} is nondegenerate, which implies $r_k = n$ in (2.3), then there exists a unique basic solution of (2.4). Since the simplex

$$C(x_{0_k}) \cap \{x \in I\!R^n \mid c_k^{\mathsf{T}}(x - x_{0_k}) \leq 1\}$$

is contained in $L(\widehat{\gamma})$ and $P_k \subset C(x_{0_k})$, the *concavity cut* $c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ is a valid cut. Figure 1 illustrates the construction.

We can deepen the concavity cut in a second step by applying cone adaptation. For this we pull the base x_{0_k} of the cone $C(x_{0_k})$ in the direction $x_{0_k} - \lambda c_k$, $\lambda \geq 0$, away from the polytope P_k, until it lies on the boundary of $L(\widehat{\gamma})$, i.e., we determine $\lambda_{0_k} \geq 0$ such that

$$x'_{0_k} := x_{0_k} - \lambda_{0_k} c_k \in \text{bd}\Big(L(\widehat{\gamma})\Big).$$

Then we consider the cone

$$C(x'_{0_k}) = x'_{0_k} + \text{cone}(u'_{1_k}, u'_{2_k}, \dots, u'_{s_k}), \tag{2.5}$$

where $u'_{1_k}, u'_{2_k}, \dots, u'_{s_k} \in I\!R^n$ with $s_k \geq r_k \geq n$ are chosen such that $C(x'_{0_k})$ is the smallest P_k-containing cone of the form (2.5). Since P_k is a polytope we can

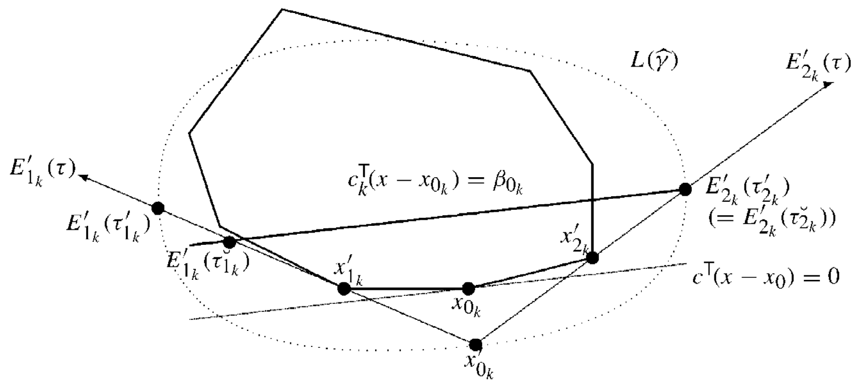

Figure 2. The cut $c_k^{\mathsf{T}}(x - x_{0_k}) \geq \beta_{0_k}$.

assume w.l.o.g. that there exist vertices $x_{1_k}, x_{2_k}, \ldots, x_{s_k}$ of P_k such that

$$u'_{i_k} = x_{i_k} - x'_{0_k} \quad \text{for } i = 1, 2, \ldots, s. \tag{2.6}$$

Furthermore, we can assume w.l.o.g. $x_{i_k} \in L(\widehat{\gamma})$ because otherwise x_{i_k} is a solution of problem (1.1) with a smaller objective value than the best solution known so far. Based on the concavity cut depicted in Figure 1 the corresponding cone $C(x'_{0_k})$ is illustrated in Figure 2.

With the cone $C(x'_{0_k})$ we can derive a valid cut that is at least equivalent to the concavity cut, i.e., it eliminates a portion of P_k that is at least as large as the portion eliminated by the concavity cut (c.f. Porembski, 2001, Theorem 3.3). The cut is derived in two steps. In the first step, using a procedure similar to that for the concavity cut, we determine the intersection points of the cone edges of $C(x'_{0_k})$ with the boundary of $L(\widehat{\gamma})$. Let $E'_{1_k}(\tau'_{1_k}), E'_{2_k}(\tau'_{2_k}), \ldots, E'_{s_k}(\tau'_{s_k})$ be these points. Then we push the hyperplane $c_k^{\mathsf{T}}(x - x_{0_k}) = 0$, which supports the polytope P_k at x_{0_k}, as far as possible forward under the condition that it still intersects all edges of $C(x'_{0_k})$ in $L(\widehat{\gamma})$. This yields a hyperplane $c_k^{\mathsf{T}}(x - x_{0_k}) = \beta_{0_k}$ with $\beta_{0_k} \geq 1$. Obviously $c_k^{\mathsf{T}}(x - x_{0_k}) \geq \beta_{0_k}$ is also a valid cut. Figure 2 shows such a cut.

However, as Figure 2 also shows, in most cases the cut $c_k^{\mathsf{T}}(x - x_{0_k}) \geq \beta_{0_k}$ is not the deepest cut possible. To get a deeper cut we determine the intersection points $E'_{1_k}(\breve{\tau}_{1_k}), E'_{2_k}(\breve{\tau}_{2_k}), \ldots, E'_{s_k}(\breve{\tau}_{s_k})$ of the edges of $C(x'_{0_k})$ with the hyperplane $c_k^{\mathsf{T}}(x - x_{0_k}) = \beta_{0_k}$ (see Figure 2) and solve the linear program

$$\begin{aligned} \text{minimize} \quad & c_k^{\mathsf{T}} d \\ \text{s.t.} \quad & d^{\mathsf{T}}(E'_{i_k}(\tau'_{i_k}) - x_{0_k}) \geq 1 \quad \text{for } i = 1, 2, \ldots, s, \\ & d^{\mathsf{T}}(E'_{i_k}(\breve{\tau}_{i_k}) - x_{0_k}) \leq 1 \quad \text{for } i = 1, 2, \ldots, s, \end{aligned} \tag{2.7}$$

i.e., we determine a valid cut $d_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ that passes through the edges of $C(x'_0)$ between $E'_{i_k}(\breve{\tau}_{i_k})$ and $E'_{i_k}(\tau'_{i_k})$ and thereby maximizes the distance from x_{0_k}

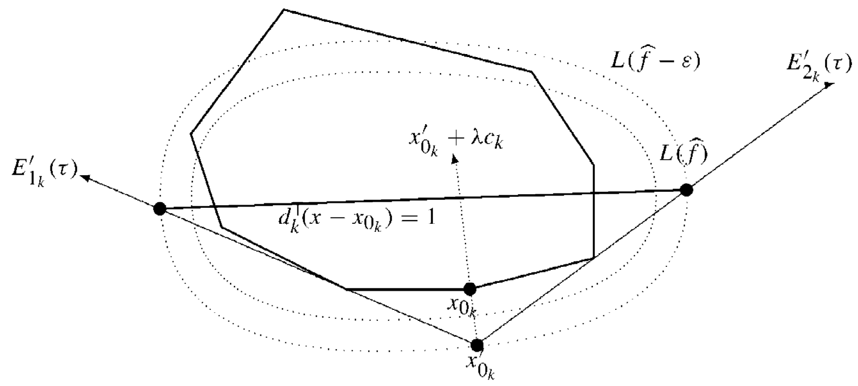

Figure 3. The cut $d_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$.

to its intersection point with the ray $x_{0_k} + \lambda c_k$, $\lambda \geq 0$. Note that a cut $d^{\mathsf{T}}(x - x_{0_k})$, where d fulfills the constraints in (2.7), intersects the ray $x_{0_k} + \lambda c_k$ in $x_{0_k} + \lambda_d c_k$ with $\lambda_d = 1/c_k^{\mathsf{T}} d$. This is illustrated in Figure 3, where $\widehat{\gamma} = \widehat{f} - \varepsilon$.

Now that we have outlined the basic concepts of a pure cutting plane algorithm based on cone adaptation, let us discuss the convergence properties of the algorithm. To this end let us, as is usual, measure the depth $\Delta(d_k)$ of the cut $d_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ as the distance from the vertex x_{0_k} to be eliminated to the hyperplane $d_k^{\mathsf{T}}(x - x_{0_k}) = 1$, i.e., $\Delta(d_k) = 1/\|d_k\|$, where $\|\cdot\|$ denotes the Euclidian norm. Then with the concepts introduced in Porembski (2001) we can prove the following.

THEOREM 2.1. *For any δ with $\delta > 0$ there exists a $\widehat{\Delta} = \widehat{\Delta}(\delta) > 0$, which is independent of the respective iteration of the cutting plane algorithm, such that the following holds: If*

$$\|x'_{0_k} - x_{0_k}\| \geq \delta \quad \textit{and} \quad \min\{\|x_{i_k} - y\| \mid y \in \mathrm{bd}(L(\widehat{\gamma}))\} \geq \delta \tag{2.8}$$

for $i = 1, 2, \ldots, s$ (see (2.5) and (2.6)), then the depth $\Delta(d_k)$ of the cut $d_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ is at least $\widehat{\Delta}$.

That a cutting plane algorithm based on cone adaptation is finitely convergent when $\varepsilon > 0$ follows with Theorem 2.1 from the following observations: First, since $f(x)$ is concave and finite on $I\!R^n$, which implies its continuity, and $L(\gamma)$ is compact for all real numbers γ, there exists a constant $\delta > 0$ such that for all $x \in P$ the distance from any point in $L(f(x))$ to $\mathrm{bd}(L(f(x) - \varepsilon))$ is at least δ. Second, since $\widehat{f}$ is the objective value of the incumbent solution we have $x_{0_k}, x_{1_k}, \ldots, x_{s_k} \in L(\widehat{f})$ (c.f. Figure 3). Hence for each iteration k the conditions of Theorem 2.1 are fulfilled. Therefore, according to Theorem 2.1, the cuts $d_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ have a depth of at least $\widehat{\Delta}$, which implies the finite convergence of the cutting plane algorithm (c.f. Horst and Tuy, 1996, Theorem V.2).

3. Finite convergence in the exact case

3.1. INTRODUCTION

As we have seen in the previous section, cutting planes derived by applying cone adaptation have at least a certain depth as long as the prescribed tolerance ε is strictly positive. However, what happens when we choose $\varepsilon = 0$? Are there also ways to ensure the finite convergence of a pure cutting plane algorithm based on these cuts without introducing enumerative elements such as facial cuts (c.f. Majthay and Whinston, 1974)?

One way to ensure finite convergence is as follows. For the following theoretical discussion let us assume that for the objective values $\widehat{f}$ of all the incumbent solutions encountered in the process of the cutting plane algorithm the concave minimization problem (1.1) fulfills the FC condition defined below, where FC stands for finite convergence.

FC *For any* x_1 *and* x_2 *lying on edges of* P *with* $\widehat{f} \leq \min\{f(x_1), f(x_2)\}$ *we have* $\text{conv}(x_1, x_2) \cap \text{bd}(L(\widehat{f})) \subseteq \{x_1, x_2\}$.

That is, a line connecting any two points lying on edges of P with objective values not smaller than $\widehat{f}$ has at most its endpoints in common with $\text{bd}(L(\widehat{f}))$. This condition implies that any face of P that is completely contained in $\text{bd}(L(\widehat{f}))$ has to be a vertex of P. The FC condition is fulfilled, for instance, when the objective function $f(x)$ is strictly concave. In general, it will be fulfilled by most concave minimization problems, even though this might be difficult to verify. However, in the last section we will show how this condition can be weakened and replaced by a local criterion. As long as this criterion is fulfilled, which has to be checked in each iteration, the finiteness and exactness of the cutting plane algorithm is ensured.

In contrast to Porembski (2001), we do not enforce the finite convergence of the cutting plane algorithm by modifying the cuts, but rather by altering how we choose the vertices of P_k with respect to which the cuts are derived. The basic idea behind the modifications is that whenever the respective local optimum is close to the boundary of the level set we look in its neighborhood for a vertex of P or for an intersection point of an edge of P with the boundary of the level set. In the first case, we can derive with respect to the identified vertex a cut for which a certain depth can always be ensured. In the second case, since one of the vertices connected by the edge is not contained in $L(\widehat{f})$, we can identify a new incumbent solution. By these modifications we ensure that after a finite number of iterations we get a 'δ-belt' around the polytope that is completely contained in the level set. Then, similar to the case where $\varepsilon > 0$, this δ-belt ensures that the cuts derived by applying cone adaptation have a certain depth. Hence the resulting cutting plane algorithm is finitely convergent.

3.2. TYPES OF VERTICES OF P_k AND VALID CUTS

To ensure the finite convergence of the cutting plane algorithm we distinguish between different types of vertices of P_k with respect to which a valid cut is derived. One is that of vertices of P_k which are also vertices of P. For these vertices the following holds.

THEOREM 3.1. *There exists a constant* $\Delta_1 > 0$, *which is independent of the respective iteration, such that the depth of all concavity cuts derived with respect to* x_0 *with* $x_0 \in \mathrm{vert}(P_k) \cap \mathrm{vert}(P)$ *is at least* Δ_1.

Proof. Let $x_0 \in \mathrm{vert}(P)$ be arbitrarily chosen, let $x_1, x_2, \ldots, x_s \in \mathrm{vert}(P)$ be its adjacent vertices and let

$$f_{x_0} := \min\{f(x_0), f(x_1), \ldots, f(x_s)\}. \tag{3.9}$$

We now derive a concavity cut $c_{x_0}^{\mathsf{T}}(x - x_0) \geq 1$ with respect to P and $L(f_{x_0})$. By construction its depth $\Delta(c_{x_0})$ is strictly positive. Since $\mathrm{vert}(P)$ is finite we also have

$$\widetilde{\Delta} := \min\{\Delta(x_0) \mid x_0 \in \mathrm{vert}(P)\} > 0. \tag{3.10}$$

Now let us consider a local optimal vertex x_{0_k} of P_k with $x_{0_k} \in \mathrm{vert}(P)$. We can assume w.l.o.g. $f(x_{0_k}) \geq \widehat{f}$, where $\widehat{f}$ is, as above, the objective value of the incumbent solution. Since $x_{0_k} \in \mathrm{vert}(P) \cap \mathrm{vert}(P_k)$, $P_k \subset P$ and $L(f(x_{0_k})) \subseteq L(\widehat{f})$, the concavity cut $c_{x_0}^{\mathsf{T}}(x - x_0) \geq 1$ is dominated by or equivalent to the corresponding concavity cut $c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ derived with respect to P_k and $L(\widehat{f})$. This implies that the cut $c_k^{\mathsf{T}}(x - x_{0_k}) \geq 1$ also has a depth of at least $\widetilde{\Delta}$. Since $\widetilde{\Delta}$ is independent of the respective iteration, this proves the proposition. □

Hence if we have a local optimal vertex x_{0_k} of P_k that is also a vertex of P, then we can be sure that the resulting concavity cut has a certain depth. This is also true for the corresponding cut derived by cone adaptation since this cut is at least equivalent to the concavity cut (Porembski, 2001, Theorem 3.3). However, if x_{0_k} is not a vertex of P, then such a general statement is not possible. The following concepts will be useful in dealing with this case.

By defining

$$V(\widehat{f}) := \{x \in \mathrm{vert}(P) \mid f(x) \geq \widehat{f}\} \tag{3.11}$$

and

$$\overline{V}(\widehat{f}) := \mathrm{vert}(P) \setminus V(\widehat{f}) \tag{3.12}$$

we partition the set $\mathrm{vert}(P)$ with respect to the objective value of the incumbent solution into two subsets: one defined by the vertices of P contained in $L(\widehat{f})$ and one defined by the vertices of P not contained in $L(\widehat{f})$. It holds:

$$\mathrm{vert}(P) = V(\widehat{f}) \cup \overline{V}(\widehat{f}) \quad \text{and} \quad V(\widehat{f}) \cap \overline{V}(\widehat{f}) = \emptyset.$$

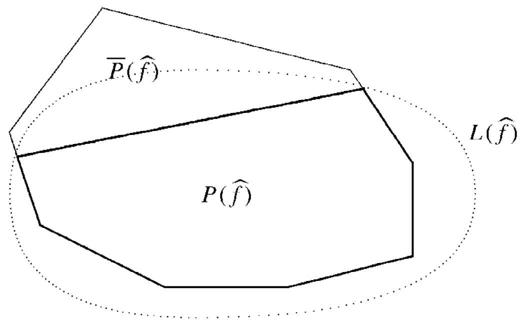

Figure 4. The Polytopes $P(\widehat{f})$ and $\overline{P}(\widehat{f})$.

Let $E(\widehat{f})$ denote the set of edges of the polytope P for which one endpoint is in $V(\widehat{f})$ and the other in $\overline{V}(\widehat{f})$. That is, $E(\widehat{f})$ is the set of edges of P that intersect $\mathrm{bd}(L(\widehat{f}))$. Because of the FC condition these intersection points are unique. Let us denote by $V_E(\widehat{f})$ the set of these intersection points. With this we define the polytopes

$$P(\widehat{f}) := \mathrm{conv}\Big(V(\widehat{f}), V_E(\widehat{f})\Big) \text{ and } \overline{P}(\widehat{f}) := \mathrm{conv}\Big(\overline{V}(\widehat{f}), V_E(\widehat{f})\Big). \qquad (3.13)$$

Figure 4 shows how the polytopes $P(\widehat{f})$ and $\overline{P}(\widehat{f})$ are obtained from the polytope P depicted in Figures 1–3. Note that $P(\widehat{f})$ and $\overline{P}(\widehat{f})$ are not necessarily disjoint.

It holds that

$$P = P(\widehat{f}) \cup \overline{P}(\widehat{f}) \quad \text{and} \quad P(\widehat{f}) \subset L(\widehat{f}). \qquad (3.14)$$

In general, we will not have $\overline{P}(\widehat{f}) \subset I\!R^n \setminus L(\widehat{f})$. But it holds that $\dim(P(\widehat{f})) = n$, and for $\overline{P}(\widehat{f})$ it follows from the FC condition that we have either $\dim(\overline{P}(\widehat{f})) = n$ or $\overline{P}(\widehat{f}) = \emptyset$. In the latter case the incumbent solution is a global optimum. In the following we discuss some properties of $P(\widehat{f})$ and $\overline{P}(\widehat{f})$.

LEMMA 3.1. *Let $g^{\mathsf{T}}x \geq \vartheta$ be a valid cut generated in a cutting plane algorithm for problem (1.1) and let $\widehat{f}$ be the objective value of the incumbent solution. Then for any $\widetilde{x} \in \overline{P}(\widehat{f}) \setminus P(\widehat{f})$ we have $g^{\mathsf{T}}\widetilde{x} > \vartheta$.*

Proof. Since $g^{\mathsf{T}}x \geq \vartheta$ is a valid cut and because of the FC condition we have

$$g^{\mathsf{T}}x' \geq \vartheta \quad \forall\, x' \in V_E(\widehat{f}) \quad \text{and} \quad g^{\mathsf{T}}x'' > \vartheta \quad \forall\, x'' \in \overline{V}(\widehat{f}). \qquad (3.15)$$

Let $\widetilde{x} \in \overline{P}(\widehat{f}) \setminus P(\widehat{f})$. It follows from the definition of $\overline{P}(\widehat{f})$ in (3.13) that there exist $\lambda_i \geq 0$ and $\mu_j \geq 0$ with $\sum_{x_i \in V_E(\widehat{f})} \lambda_i + \sum_{y_j \in \overline{V}(\widehat{f})} \mu_j = 1$ such that

$$\widetilde{x} = \sum_{x'_i \in V_E(\widehat{f})} \lambda_i x'_i + \sum_{x''_j \in \overline{V}(\widehat{f})} \mu_j x''_j. \qquad (3.16)$$

By assumption we have $\widetilde{x} \notin P(\widehat{f})$. Hence there exists at least one vertex x''_{i_0} in $\overline{V}(\widehat{f})$ such that in (3.16) for the corresponding μ_{i_0} the strict inequality $\mu_{i_0} > 0$ holds. Because of (3.15) we therefore have $g^{\mathsf{T}}\widetilde{x} > \vartheta$. □

That is, no point $\widetilde{x} \in \overline{P}(\widehat{f}) \setminus P(\widehat{f})$ can be eliminated or 'touched' by a valid cut. Now as above let P_k denote the polytope we face in the kth iteration of the cutting plane algorithm. The following holds.

PROPOSITION 3.1. *If* $x_0 \in \mathrm{vert}(P_k)$ *with* $f(x_0) \geq \widehat{f}$, *then* $x_0 \in P(\widehat{f})$.

Proof. Since $P_k \subset P$ we have $P_k \subset P(\widehat{f}) \cup \overline{P}(\widehat{f})$. Let x_0 be a vertex of P_k with $f(x_0) \geq \widehat{f}$. Then we either have $x_0 \in \mathrm{vert}(P)$ or there exists a valid cut $d_j^{\mathsf{T}}(x - x_{0_j}) \geq 1$, derived in a previous iteration of the cutting plane algorithm, with $d_j^{\mathsf{T}}(x_0 - x_{0_j}) = 1$. In the first case we have $x_0 \in V(\widehat{f})$, which implies $x_0 \in P(\widehat{f})$, and in the second case it follows from Lemma 3.1 that we have $x_0 \in P(\widehat{f})$. □

Therefore, any vertex of P_k with an objective value at least as large as that of the incumbent solution is contained in the polytope $P(\widehat{f})$. For what follows let us denote by $V_=(\widehat{f})$ the set of vertices in $V(\widehat{f})$ lying on $\mathrm{bd}(L(\widehat{f}))$, i.e.,

$$V_=(\widehat{f}) := \{x \in V(\widehat{f}) \mid f(x) = \widehat{f}\}. \tag{3.17}$$

Furthermore, let $B(x, \varrho)$ be an open ball around x with radius ϱ, i.e. $B(x, \varrho) := \{y \in I\!R^n \mid \|y - x\| < \varrho\}$. The following holds.

PROPOSITION 3.2. *For any constant* $\varrho > 0$ *there exists* $\Delta_2(\widehat{f}, \varrho) > 0$ *such that if* $x \in P(\widehat{f})$ *and* $x \notin B(y, \varrho)$ *for all* $y \in V_=(\widehat{f}) \cup V_E(\widehat{f})$, *then the distance from* x *to* $\mathrm{bd}(L(\widehat{f}))$ *is at least* $\Delta_2(\widehat{f}, \varrho)$.

Proof. By construction we have $P(\widehat{f}) \subset L(\widehat{f})$. Because of the FC condition the only points in $P(\widehat{f})$ lying on $\mathrm{bd}(L(\widehat{f}))$ are those in $V_=(\widehat{f}) \cup V_E(\widehat{f})$. Therefore, for

$$Q_\varrho(\widehat{f}) := P(\widehat{f}) \setminus \left\{ \bigcup_{x'_i \in V_=(\widehat{f})} B(x'_i, \varrho) \;\; \cup \bigcup_{x''_i \in V_E(\widehat{f})} B(x''_i, \varrho) \right\}$$

we have $Q_\varrho(\widehat{f}) \subset L(\widehat{f})$ and $Q_\varrho(\widehat{f}) \cap \mathrm{bd}(L(\widehat{f})) = \emptyset$ for all $\varrho > 0$. Since Q_ϱ is compact by construction and $L(\widehat{f})$ by assumption, there exists a constant $\Delta_2(\widehat{f}, \varrho) > 0$ such that

$$\min \left\{ \|x - y\| \;\middle|\; x \in Q_\varrho(\widehat{f}),\; y \in \mathrm{bd}(L(\widehat{f})) \right\} \geq \Delta_2(\widehat{f}, \varrho),$$

which proves the proposition. □

Note that if the FC condition does not hold then even facets of $P(\widehat{f})$ might be contained in $\mathrm{bd}(L(\widehat{f}))$ and a statement as in Proposition 3.2 would not be possible.

With Propositions 3.1 and 3.2 it is now easy to prove the following theorem, which will provide us the basis for the modified cutting plane algorithm.

THEOREM 3.2. *Let $x_{0_k} \in \mathrm{vert}(P_k)$ with $f(x_{0_k}) \geq \widehat{f}$. Furthermore, let $\varrho > 0$ be prescribed and let $\Delta_2(\widehat{f}, \varrho)$ be defined according to Proposition 3.2. Then it holds that either the distance from x_{0_k} to $\mathrm{bd}(L(\widehat{f}))$ is at least $\Delta_2(\widehat{f}, \varrho)$ or there exists $y \in V_{=}(\widehat{f}) \cup V_E(\widehat{f})$ such that $x_{0_k} \in B(y, \varrho)$.*

Proof. Proposition 3.1 implies $x_{0_k} \in P(\widehat{f})$. With this the theorem follows immediately from Proposition 3.2. □

If for a vertex x_{0_k} of P_k with $f(x_{0_k}) \geq \widehat{f}$ the distance to $\mathrm{bd}(L(\widehat{f}))$ is less than $\Delta_2(\widehat{f}, \varrho)$, then, according to Theorem 3.2, there exists $y \in V_{=}(\widehat{f}) \cup V_E(\widehat{f})$ such that $x_{0_k} \in B(y, \varrho)$. If $y \in V_E(\widehat{f})$, then there exists a y-containing edge of P. This edge connects two vertices, say y_1 and y_2 of P. Then it follows from the definition of $V_E(\widehat{f})$ that we have w.l.o.g. $f(y_1) \geq \widehat{f}$ and $f(y_2) < \widehat{f}$. That is, we have identified a vertex of P with a smaller objective value than the incumbent solution.

3.3. MODIFICATION OF THE CUTTING PLANE ALGORITHM

In this subsection we use the observations just made to construct a finite and exact cutting plane algorithm for problem (1.1). In the previous subsection we considered the objective value $\widehat{f}$ of the incumbent solution as constant. However, in the process of a cutting plane algorithm this value and the corresponding incumbent solution will usually change several times. Hence in what follows we will, whenever necessary to omit misunderstandings, denote by $\widehat{f_k}$ the objective value of the incumbent solution in the kth iteration of the cutting plane algorithm. For the corresponding incumbent solutions, the following holds true.

LEMMA 3.2. *In a cutting plane algorithm based on the search-and-cut scheme described in the first section the incumbent solution is always a vertex of P.*

Proof. We prove Lemma 3.2 by induction in k. Clearly, the first incumbent solution is a vertex of P. Hence for $k = 1$ the assertion is true. Let the assertion hold for the first $k - 1$ iterations. We now have to prove that it also holds for the kth iteration.

If $f(x_{0_k}) \geq \widehat{f}_{k-1}$, then the incumbent solution of the $(k-1)$th iteration remains the incumbent solution in the kth iteration, i.e., $\widehat{f_k} = \widehat{f}_{k-1}$. Hence it is a vertex of P. Suppose that we have $f(x_{0_k}) < \widehat{f}_{k-1}$ and assume that $x_{0_k} \in \mathrm{vert}(P_k) \setminus \mathrm{vert}(P)$. Therefore, there exists a valid cut $d_j^{\mathsf{T}}(x - x_{0_j}) \geq 1$ derived in a previous iteration of the cutting plane algorithm with $d_j^{\mathsf{T}}(x_{0_k} - x_{0_j}) = 1$ and it holds that $\widehat{f_j} \geq \widehat{f}_{k-1}$. Since $d_j^{\mathsf{T}}(x - x_{0_j}) \geq 1$ is a valid cut with respect to P_j and $\widehat{f_j}$ we have $f(x) \geq \widehat{f_j} \geq \widehat{f}_{k-1}$ for all $x \in P_j$ with $d_j^{\mathsf{T}}(x - x_{0_j}) \leq 1$. Because of $P_k \subset P_j$ and $d_j^{\mathsf{T}}(x_{0_k} - x_{0_j}) = 1$

this also implies that $f(x_{0_k}) \geq \widehat{f}_{k-1}$, which is in contradiction to the assumption $f(x_{0_k}) < \widehat{f}_{k-1}$. Hence we have $x_{0_k} \in \mathrm{vert}(P_k) \cap \mathrm{vert}(P)$, which proves Lemma 3.2. □

Since $\mathrm{vert}(P)$ is finite we can derive the following assertion from Lemma 3.2.

COROLLARY 3.1. *The set $\mathcal{F} = \{f_1, f_2, \ldots, f_z\}$ of all objective values which can be attained by $\widehat{f}$ is finite.*

In the following we use the concepts introduced above to modify the choice of a vertex with respect to which a valid cut is derived in such a way that the finite convergence of the cutting plane algorithm can be ensured. To this end we choose the radius ϱ of the open balls around the points y in $V_=(\widehat{f}) \cup V_E(\widehat{f})$ (c.f. Proposition 3.2) such that

$$0 < \varrho < \Delta_1 \tag{3.18}$$

holds, where Δ_1 is defined according to Theorem 3.1 as a lower bound for the depth of concavity cuts derived with respect to vertices of P_k that are also vertices of P. With this we define

$$\delta := \min\{\Delta_2(f, \varrho) \mid f \in \mathcal{F}\} > 0, \tag{3.19}$$

where according to Proposition 3.2 $\Delta_2(f, \varrho)$ is a lower bound for the distance from any x with $x \notin B(y, \varrho)$ for all $y \in V_=(\widehat{f}) \cup V_E(\widehat{f})$ to the boundary of $L(\widehat{f})$, and $\mathcal{F}$ is defined according to Corollary 3.1. Note that the existence of δ in (3.19) is ensured by the finiteness of $\mathcal{F}$ (see Corollary 3.1). How to deal with the fact that in practice Δ_1 and $V_=(\widehat{f}) \cup V_E(\widehat{f})$ are not explicitly known is discussed in the next section.

Let us now consider the cut phase of the kth iteration. As above let x_{0_k} be the local optimum to be eliminated and let $x_{1_k}, x_{2_k}, \ldots, x_{s_k}$ be the vertices of P_k used to define the directions of the cone $C(x'_{0_k}) = x'_{0_k} + \mathrm{cone}(u'_{1_k}, u'_{2_k}, \ldots, u'_{s_k})$ (see (2.5) and (2.6)). If $x_{i_k} \notin B(y, \varrho)$ for all $y \in V_=(\widehat{f}) \cup V_E(\widehat{f})$ and $i = 0, 1, 2, \ldots, s$, then we derive a valid cut in the usual way. However, if there exists $\widehat{y} \in V_=(\widehat{f}) \cup V_E(\widehat{f})$ with $x_{i'_k} \in B(\widehat{y}, \varrho)$ for some $x_{i'_k} \in \{x_{0_k}, x_{1_k}, \ldots, x_{s_k}\}$, then a modification is required.

To this end we distinguish between the cases $\widehat{y} \in V_=(\widehat{f})$ and $\widehat{y} \in V_E(\widehat{f})$. In the first case we derive a concavity cut with respect to $\widehat{y}$ and go to the next iteration. Note that because $\widehat{y} \in \mathrm{bd}(L(\widehat{f}))$ it is not possible to deepen this cut by applying cone adaptation. In the second case we identify a vertex $\widetilde{x}$ of P with $f(\widetilde{x}) < \widehat{f}$ as described in the previous subsection. Then we return to the search phase and, starting at $\widetilde{x}$, which is also a vertex of P_k, we identify a new local optimal vertex $\widetilde{x}_{0_k}$ of P_k. The following holds:

THEOREM 3.3. *By modifying the choice of vertices with respect to which cuts are derived as described above we ensure the finite convergence of the corresponding cutting plane algorithm.*

Proof. Using the notation above we have to distinguish among three cases.

(1) Suppose we have $x_{i'_k} \in B(\widehat{y}, \varrho)$ with $\widehat{y} \in V_=(\widehat{f})$ for some $i' \in \{0, 1, \dots, s\}$. Then we derive a concavity cut with respect to $\widehat{y}$. Since according to Theorem 3.1 this cut has a depth of at least Δ_1 and $x_{i'_k} \in B(\widehat{y}, \varrho)$ with $\varrho < \Delta_1$ (see (3.18)), it also eliminates $x_{i'_k}$. Furthermore, because of the special choice of ϱ in (3.18) we will not encounter $\widehat{y}$ in a subsequent iteration of the cutting plane algorithm.

(2) Suppose we have $x_{i'_k} \in B(\widehat{y}, \varrho)$ with $\widehat{y} \in V_E(\widehat{f})$ for some $i' \in \{0, 1, \dots, s\}$. Then we identify the corresponding $\widetilde{x} \in \text{vert}(P) \cap \text{vert}(P_k)$ with $f(\widetilde{x}) < \widehat{f}$. Starting at $\widetilde{x}$ we find a new incumbent solution $\widehat{x}$ with $f(\widehat{x}) \le f(\widetilde{x}) < \widehat{f}$. Hence $\widetilde{x}$ will not be identified in a subsequent iteration as a solution with a smaller objective value than the incumbent solution.

(3) If $x_{i_k} \notin B(y, \varrho)$ for all $y \in V_=(\widehat{f}) \cup V_E(\widehat{f})$ and $i = 0, 1, \dots, s$ because of (3.19) and Theorem 3.2 we have $\|x_{0_k} - x'_{0_k}\| \ge \delta$ and $\min\{\|x_{i_k} - y\| \mid y \in \text{bd}(L(\widehat{f}))\} \ge \delta$ for $i = 1, 2, \dots, s$. Therefore, according to Theorem 2.1, we can derive a valid cut with a depth of at least $\widehat{\Delta}$ by applying cone adaptation.

Since $\text{vert}(P)$ is finite we have ensured that after a finite number of iterations we always face the last case. Since the cuts derived in the last case have a depth of at least $\widehat{\Delta}$ and $\widehat{\Delta}$ is independent of the respective iteration, the cutting plane algorithm terminates after a finite number of iterations (c.f. Horst and Tuy, 1996, Theorem V.2). □

4. Weakening of conditions and implementation

In the previous section we outlined a modification of the cutting plane algorithm based on cone adaptation that ensures finite convergence of the algorithm in the case where $\varepsilon = 0$. To this end we have to choose in advance the radius ϱ of the open balls $B(\varrho, y)$, $y \in V_=(\widehat{f}) \cup V_E(\widehat{f})$, such that $\varrho < \Delta_1$ holds (see (3.18)), where Δ_1 is a lower bound for the depth of concavity cuts derived with respect to vertices of P. However, Δ_1 is not explicitly known. Furthermore, the points in $V_=(\widehat{f}) \cup V_E(\widehat{f})$ for $\widehat{f} \in \mathcal{F}$ are also not explicitly known. To overcome these problems we have to choose ϱ in an adaptive way which might result in different values of ϱ at each iteration and for each y in $V_=(\widehat{f_k}) \cup V_E(\widehat{f_k})$, i.e. $\varrho = \varrho_k(y)$. To this end we have to weaken the conditions which must be fulfilled by ϱ without endangering the finite convergence of the corresponding cutting plane algorithm.

PROPOSITION 4.1. *Let $\varrho_k(y) > 0$ with $y \in V_=(\widehat{f_k}) \cup V_E(\widehat{f_k})$. Then the modifications of the cutting plane algorithm outlined in Subsection 3.3 lead to a finite cutting plane algorithm as long as the following holds for $\varrho_k(y)$:*

(1) If $x_{0_k} \in B(\widehat{y}, \varrho_k(\widehat{y}))$ for some $\widehat{y} \in V_=(\widehat{f_k})$ and a concavity cut is derived with respect to $\widehat{y}$, then $x_{0_j} \notin B(\widehat{y}, \varrho_j(\widehat{y}))$ for all $j > k$.

(2) There exists a constant $\varrho^L > 0$ with $\varrho^L \le \varrho_k(y)$ for all $k \ge 1$ and all $y \in V_=(\widehat{f_k}) \cup V_E(\widehat{f_k})$.

Proof. The first condition ensures that whenever a vertex y of P with $y \in V_{=}(\widehat{f_k})$ has been eliminated by a concavity cut in the kth iteration we will not encounter it in the context of concavity cuts at a later iteration (see (3.18)). The second condition ensures that there exists a constant $\delta' > 0$ which is independent of the respective iteration such that for all $x \in P(\widehat{f_k})$ that are not contained in an open ball $x \notin B(y, \varrho_k(y))$ for any $y \in V_{=}(\widehat{f_k}) \cup V_E(\widehat{f_k})$ the distance from x to $\mathrm{bd}(L(\widehat{f_k}))$ is at least δ'. Using arguments similar to those in the proof of Theorem 3.3 we can see that the corresponding cutting plane algorithm remains finitely convergent. □

In the modified cutting plane algorithm we have to determine whether a vertex x of P_k is contained in a φ-neighborhood of a vertex y of P lying on the boundary of $L(\widehat{f_k})$ or in a φ-neighborhood of an intersection point of an edge of P with the boundary of $L(\widehat{f_k})$, i.e., $y \in V_{=}(\widehat{f_k}) \cup V_E(\widehat{f_k})$. To this end we make use of the following well-known properties of vertices and edges of $P = \{x \in I\!R^n \mid Ax \leq b\}$, where we assume w.l.o.g. that (A, b) is of full rank and that $Ax \leq b$ contains no redundant constraints:

$\widetilde{y}$ is a vertex of P if and only if $Ax \leq b$ can be uniquely partitioned into subsystems $A_1 x \leq b_1$ and $A_2 x \leq b_2$ with $\mathrm{rank}(A_1) = n$ such that

$$A_1 \widetilde{y} = b_1,$$
$$A_2 \widetilde{y} < b_2,$$

where $A_1 \in I\!R^{t \times n}$, $b_1 \in I\!R^t$, $A_2 \in I\!R^{(m-t) \times n}$, $b_2 \in I\!R^{m-t}$ and $t \geq n$. Accordingly, $\widetilde{y}$ lies on an edge of P if and only if in the partition above $\mathrm{rank}(A_1) = n - 1$ and $t = n - 1$ holds.

Based on this we can see that for a sufficiently small $\varrho' > 0$ there exists a constant $\varphi > 0$ such that

$$x \in B(\widetilde{y}, \varrho') \cap P \qquad \Rightarrow \qquad \begin{array}{rcl} b_1 - \varphi e \leq & A_1 x & \leq b_1, \\ & A_2 x & < b_2 - \varphi e, \end{array} \tag{4.20}$$

holds, where $e = (1, \ldots, 1)^{\mathsf{T}}$. Conversely, for a given $\varphi > 0$ there exists a $\varrho'' > 0$ such that for all $x \in I\!R^n$ with

$$\begin{array}{rcl} b_1 - \varphi e \leq & A_1 x & \leq b_1, \\ & A_2 x & < b_2 - \varphi e \end{array} \qquad \Rightarrow \qquad x \in B(\widetilde{y}, \varrho'') \cap P. \tag{4.21}$$

Note that in (4.21) for a given φ there exists a uniquely determined lower bound $\widetilde{\varrho}(\widetilde{y}, \varphi)$ for ϱ''. Obviously $\widetilde{\varrho}(\widetilde{y}, \varphi)$ is non-increasing for decreasing values of φ and it holds that

$$0 < \varrho' \leq \widetilde{\varrho}(\widetilde{y}, \varphi). \tag{4.22}$$

Based on these observations we can implement the modifications of the cutting plane algorithm proposed in Subsection 3.3 as follows. As above let x_{0_k} be the local optimum to be eliminated and let $x_{1_k}, x_{2_k}, \dots, x_{s_k}$ be the vertices of P_k used to define the directions of the cone $C(x'_{0_k}) = x'_{0_k} + \text{cone}(u'_{1_k}, u'_{2_k}, \dots, u'_{s_k})$ (see (2.5) and (2.6)) and let $\mathcal{I}_k := \{0_k, 1_k, \dots, s_k\}$ and $\mathcal{J}_k = \emptyset$. For $i_k \in \mathcal{I}_k$ we first partition the system $Ax \le b$ such that

$$\begin{aligned} b_{1_{i_k}} - \varphi_k e \;\le\; A_{1_{i_k}} x_{i_k}, \\ A_{2_{i_k}} x_{i_k} \;<\; b_{2_{i_k}} - \varphi_k e \end{aligned} \tag{4.23}$$

holds, where φ_k for $k = 1$ is prechosen and strictly positive and for $k > 1$ is obtained from φ_{k-1} as described below. Next we have to distinguish among the following cases:

(1) If $\text{rank}(A_{1_{i'_k}}) = \text{rank}(A_{1_{i'_k}}, b_{1_{i'_k}}) = n$ for some $i'_k \in \mathcal{I}_k \setminus \mathcal{J}_k$, then there exists a uniquely determined $y_{i'_k} \in I\!R^n$ with $A_{1_{i'_k}} y_{i'_k} = b_{1_{i'_k}}$.

(a) If it also holds that $A_{2_{i'_k}} y_{i'_k} < b_{2_{i'_k}}$, then $y_{i'_k}$ is a vertex of P. We then face one of the following cases:

(i) If $f(y_{i'_k}) < \widehat{f_k}$, then we set $\widehat{f_k} := f(y_{i'_k})$. Note that $y_{i'_k}$ is also a vertex of P_k. We go back to the search phase of the kth iteration and determine, starting at $y_{i'_k}$, a new local optimum.

(ii) If $f(y_{i'_k}) > \widehat{f_k}$ or $f(y_{i'_k}) = \widehat{f_k}$, where the latter case implies that $y_{i'_k} \in V_=(\widehat{f_k})$, we have to distinguish between $y_{i'_k} \in P_k$ and $y_{i'_k} \notin P_k$:

(A) If $y_{i'_k} \in P_k$, then $y_{i'_k}$ is also a vertex of P_k. Hence in the case of $f(y_{i'_k}) > \widehat{f_k}$ we derive a cutting plane with respect to $y_{i'_k}$ by applying cone adaptation and in the case of $f(y_{i'_k}) = \widehat{f_k}$ by applying a concavity cut. Then we set $\varphi_{k+1} := \varphi_k$ and go to the next iteration of the cutting plane algorithm.

(B) If $y_{i'_k} \notin P_k$, then $y_{i'_k}$ has already been eliminated by a concavity cut. Therefore, to fulfill condition 1. of Proposition 4.1 we have to decrease φ_k in such a way that $b_{1_{i'_k}} - \varphi_k e \not\le A_{1_{i'_k}} x_{i'_k}$. This can be done, for instance, by setting

$$\varphi_k := \tfrac{1}{\theta} \max_{\ell=1}^{t} \Big(\beta_{1_{i'_k}\ell} - a^{\mathsf{T}}_{1_{i'_k}\ell} x_{i'_k} \Big), \tag{4.24}$$

where $\theta > 1$ is appropriately prechosen and $a^{\mathsf{T}}_{1_{i'_k}\ell} x \le \beta_{1_{i'_k}\ell}$ denotes the ℓth inequality in $A_{1_{i'_k}} x \le b_{1_{i'_k}}$. Note that since $y_{i'_k}$ has already been eliminated by a valid cut there exists an index ℓ_0 such that $a^{\mathsf{T}}_{1_{i'_k}\ell_0} x_{i'_k} < \beta_{1_{i'_k}\ell_0}$, i.e. in (4.24) we have $\varphi_k > 0$. With the updated φ_k we determine the corresponding partition (4.23) of $Ax \le b$ and examine it once again.

(b) If $A_{2_{i'_k}} x_{i'_k} \not\leq b_{2_{i'_k}}$, then $y_{i'_k}$ is a basic solution but not a feasible one, i.e. $y_{i'_k}$ is a pseudovertex of P (c.f. Porembski, 1999). In this case we decrease φ_k, for instance, by setting $\varphi_k := \varphi_k/2$. With the updated φ_k we then determine, as above, the corresponding partition (4.23) of $Ax \leq b$ and examine it once again.

(2) If $\mathrm{rank}(A_{1_{i'_k}}) = \mathrm{rank}(A_{1_{i'_k}}, b_{1_{i'_k}}) = n - 1$ for some $i'_k \in \mathcal{I}_k \setminus \mathcal{J}_k$, then $L_{i'_k} := \{y \in I\!R^n \mid A_{1_{i'_k}} y = b_{1_{i'_k}}\}$ is a line in $I\!R^n$, i.e., $\dim(L_{i'_k}) = 1$. We now have to distinguish between $\dim(L_{i'_k} \cap P) = 1$ and $\dim(L_{i'_k} \cap P) < 1$.

(a) If $\dim(L_{i'_k} \cap P) = 1$, then $L_{i'_k}$ contains an edge of P. We then determine the vertices of P, say $y_{1_{i'_k}}$ and $y_{2_{i'_k}}$, connected by this edge, i.e. the edge can be described by $\mathrm{conv}(y_{1_{i'_k}}, y_{2_{i'_k}})$.

(i) If w.l.o.g. $f(y_{1_{i'_k}}) < \widehat{f_k}$, then $y_{1_{i'_k}} \in P_k$ and we have identified a new incumbent solution. Hence we set $\widehat{f_k} := f(y_{1_{i'_k}})$ and go back to the search phase of the respective iteration to identify a new local optimum starting at $y_{1_{i'_k}}$.

(ii) If $f(y_{1_{i'_k}}), f(y_{2_{i'_k}}) \geq \widehat{f_k}$, then we determine the point $\widehat{y}_{i'_k} \in L_{i'_k}$ closest to $x_{i'_k}$ by orthogonal projection. We then have to distinguish between two cases:

(A) If we have $\widehat{y}_{i'_k} \in \mathrm{conv}(y_{1_{i'_k}}, y_{2_{i'_k}})$, then there exists no point $y \in \mathrm{conv}(y_{1_{i'_k}}, y_{2_{i'_k}})$ with $y \in V_{=}(\widehat{f_k})$ and $x_{i'_k} \in B(y, \widetilde{\varrho}(y, \varphi_k))$, because otherwise we would have case 1 for i'_k. We set $\mathcal{J}_k := \mathcal{J}_k \cup \{i'_k\}$. If $\mathcal{I}_k \setminus \mathcal{J}_k = \emptyset$ then we derive a cutting plane with respect x_{0_k} by applying cone adaptation, we set $\varphi_{k+1} := \varphi_k$ and go to the next iteration. Otherwise we examine the remaining indices in $\mathcal{I}_k \setminus \mathcal{J}_k$.

(B) If $\widehat{y}_{i'_k} \notin \mathrm{conv}(y_{1_{i'_k}}, y_{2_{i'_k}})$, a situation that might occur for $n \geq 3$, we have to decrease φ_k because there exists a point $\widetilde{x}$ on the line $L_{i'_k}$ with $\widetilde{x} \notin P$ and $\widetilde{x} \notin B(y_{\ell_k}, \widetilde{\varrho}(y_{\ell_k}, \varphi_k))$ for $\ell = 1, 2$, but for which $x_{i'_k} \in B(\widetilde{x}, \widetilde{\varrho}(\widetilde{x}, \varphi_k))$ holds. This can be done, for instance, by setting $\varphi_k := \varphi_k/2$. We then examine partition (4.23) for the updated φ_k.

(b) If $\dim(L_{i'_k} \cap P) < 1$, then we decrease φ_k, as above, by setting $\varphi_k := \varphi_k/2$ and examine partition (4.23) for the updated φ_k.

(3) If $n - 1 \leq \mathrm{rank}(A_{1_{i'_k}}) < \mathrm{rank}(A_{1_{i'_k}}, b_{1_{i'_k}})$ for some $i'_k \in \mathcal{I}_k \setminus \mathcal{J}_k$, then $A_{1_{i'_k}} x = b_{1_{i'_k}}$ is unsolvable. Hence φ_k has to be decreased. Since $A_{1_{i'_k}} x_{i'_k} \leq b_{1_{i'_k}}$ and $A_{1_{i'_k}} x_{i'_k} \neq b_{1_{i'_k}}$ this can be done as in (4.24). We then examine the corresponding partition (4.23) for the updated φ_k.

(4) If $\text{rank}(A_{1_{i'_k}}) < n-1$ for some $i'_k \in \mathcal{I}_k \setminus \mathcal{J}_k$, then we have

$$x_{i'_k} \notin B(y, \widetilde{\varrho}(y, \varphi_k)) \quad \text{for all } y \in V_=(\widehat{f_k}) \cup V_E(\widehat{f_k}). \tag{4.25}$$

Hence in this case we set $\mathcal{J}_k := \mathcal{J}_k \cup \{i'_k\}$. If $\mathcal{I}_k \setminus \mathcal{J}_k = \emptyset$ then we derive a cutting plane with respect x_{0_k} by applying cone adaptation, we set $\varphi_{k+1} := \varphi_k$ and go to the next iteration. Otherwise we examine the remaining indices in $\mathcal{I}_k \setminus \mathcal{J}_k$.

Only in cases 1.a.ii.A, 2.a.ii.A and 4. do we derive cutting planes. In the first case we have identified a vertex $y_{i'_k}$ of P that is still a vertex of P_k. Note that in contrast to the scheme discussed in the previous section we switch from x_{0_k} to $y_{i'_k}$ to derive a cutting plane even when $y_{i'_k} \notin V_=(\widehat{f_k})$. However, because of Theorem 3.1 this does not have any impact on the finite convergence of the cutting plane algorithm. In the second case we have verified for the last index i'_k examined that there is no $y \in V_=(\widehat{f_k})$, which follows from $\text{rank}(A_{1_{i'_k}}) = n-1$, and no point $y \in V_E(\widehat{f_k})$ with $x_{i'_k} \in B(y, \widetilde{\varrho}(y, \varphi_k))$. In the third case we have verified for the last index i'_k examined that there is no vertex y of P or a point y lying on an edge of P with $x_{i'_k} \in B(y, \widetilde{\varrho}(y, \varphi_k))$, which follows from $\text{rank}(A_{1_{i'_k}}) < n-1$. Therefore, in the second and third case, as long as there exists $\varrho^L > 0$ with $\widetilde{\varrho}(y, \varphi_k) \geq \varrho^L$ for all $k \geq 1$, $y \in V_=(\widehat{f}) \cup V_E(\widehat{f})$ and $\widehat{f} \in \mathcal{F}$, because of Theorem 3.2 we can guarantee a certain depth for the resulting cut.

In the cases 1.a.i and 2.a.i we identify a vertex of P_k with an objective value smaller than the incumbent solution. Hence we have a new incumbent solution and when we perform our search for a new local optimum at $y_{i'_k}$ we end up with a local optimum that has an objective value smaller than the previous incumbent solution. Note that since $\text{vert}(P)$ is finite we will encounter these cases only a finite number of times.

In all other cases we decrease φ_k in such a way that the number of constraints in $A_{1_{i'_k}} x \leq b_{1_{i'_k}}$ is reduced by at least one. Hence after a finite number of updates of φ_k we face either case 1.a.ii.A or case 2.a.ii or case 4., i.e. we derive a cutting plane and go to the next iteration. The following holds:

THEOREM 4.1. *By implementing the concepts of the previous section as described above we can ensure the finite convergence of the corresponding cutting plane algorithm.*

Proof. We prove Theorem 4.1 by showing that the conditions of Proposition 4.1 are fulfilled. To this end we set $\varrho_k(y) := \widetilde{\varrho}(y, \varphi_k)$ (see (4.22)).

That the first condition is fulfilled follows from the way we decrease φ_k above (c.f. case 1.a.ii.A). That the second condition is also fulfilled follows from the fact that there exists a constant $\widehat{\varrho} > 0$ such that

(1) For all, not necessarily feasible, basic solutions y of $Ax \leq b$, i.e. all vertices and pseudovertices of P, the ball $B(y, \widehat{\varrho})$ does not contain any other basic solution.

(2) For all z lying on a line defined by a subsystem of $Ax \leq b$ but not lying in $B(y, \widehat{\varrho})$ for any basic solution y of $Ax \leq b$, the ball $B(z, \widehat{\varrho})$ is not intersected by any other line defined by a subsystem of $Ax \leq b$.

It follows from the way we decrease φ_k that after a finite number of updates we will have

$$\min\{\Delta_1, \widehat{\varrho}\} \;\geq\; \widetilde{\varrho}(y, \varphi_k) \;>\; 0 \qquad (4.26)$$

for all $y \in V_=(\widehat{f_k}) \cup V_E(\widehat{f})$ and $\widehat{f} \in \mathcal{F}$. However, as soon as (4.26) holds we stop updating φ_k, i.e. $\varphi_{k+i} = \varphi_k$ for all $i \geq 0$. Therefore, there exists $\varrho^L > 0$ such that condition 2. of Proposition 4.1 holds. □

As stated in Subsection 3.1, the concave minimization problem has to fulfill the FC condition to ensure the finiteness and exactness of the cutting plane algorithm. This is because the FC condition ensures that only vertices of $P(\widehat{f})$ lie on $\mathrm{bd}(L(\widehat{f}))$, so that Proposition 3.2 holds. Proposition 3.2 ensures that when we apply cone adaptation in cases 2.a.ii.A and 4. the resulting cut has at least a certain depth. Theorem 3.1 ensures that the cut derived in case 1.a.ii by applying cone adaptation has a certain depth.

However, we can also ensure that the cuts derived in cases 2.a.ii.A and 4. have a certain depth if for each of these cuts the conditions in Theorem 2.1 hold. If this is the case for all but a finite number of these cuts, then the cutting plane algorithm is still finite and exact, even though the FC condition might not be fulfilled. Therefore, if we are not sure whether or not the FC condition is fulfilled, then we can proceed as follows: We choose a sufficiently small δ with $\delta > 0$ and then in cases 2.a.ii.A and 4 whenever we are going to derive a cutting plane by cone adaptation we check whether

$$\min\{\|x_{i_k} - y\| \mid y \in \mathrm{bd}(L(\widehat{f}))\} \geq \delta \qquad \text{for all } i_k \in \mathcal{I}_k \qquad (4.27)$$

(c.f. Theorem 2.1). If this is the case then we derive the cut; otherwise the finiteness of the cutting plane algorithm is no longer ensured. A further modification is to allow a finite number of times the derivation of cuts that do not fulfill the condition (4.27). Clearly, these modifications have no impact on the finiteness or exactness of the cutting plane algorithm.

5. Acknowledgements

The author is grateful to two anonymous referees for their helpful comments, which led to improvements in the paper.

References

Al-Khayyal, F.A. and Sherali, H.D. (2000), On Finitely Terminating Branch-and-Bound Algorithms for Some Global Optimization Problems, *SIAM Journal on Optimization* 10, 1049–1057.

Benson, H.P. (1995), Concave Minimization: Theory, Applications and Algorithms, In: *Handbook of Global Optimization*, Horst, R. and Pardalos, P.M. (eds.), Kluwer, Dordrecht, pp. 43–148.

Benson, H.P. (1996), Deterministic Algorithms for Constrained Concave Minimization: A Unified Critical Survey, *Naval Research Logistics* 43, 765–795.

Benson, H.P. (1999), Generalized γ-Valid Cut Procedure for Concave Minimization, *Journal of Optimization Theory and Applications* 102, 289–298.

Horst, R. and Tuy, H. (1996), *Global Optimization (Deterministic Approaches)*, 3rd edition, Springer, Berlin.

Locatelli, M. and Thoai, N.V. (2000), Finite Exact Branch-and-Bound Algorithms for Concave Minimization over Polytopes, *Journal of Global Optimization* 18, 107–128.

Majthay, A.J. and Whinston, A. (1974), Quasi-Concave Minimization Subject to Linear Constraints, *Discrete Mathematics* 9, 35–59.

Mangasarian, O.L. (1969), *Nonlinear Programming*, McGraw-Hill, New York.

Porembski, M. (1996), *Cutting Planes for Concave Minimization and Related Optimization Problems*, Doctoral Thesis, University of Marburg (in German).

Porembski, M. (1999), How to Extend the Concept of Convexity Cuts to Derive Deeper Cutting Planes, *Journal of Global Optimization* 15, 371–404.

Porembski, M. (2001), Finitely Convergent Cutting Planes for Concave Minimization, *Journal of Global Optimization* 20, 113–136.

Shectman, J.P. and Sahinidis, N.V. (1998), A Finite Algorithm for Global Minimization of Separable Concave Programs, *Journal of Global Optimization* 12, 1–36.

Tuy, H. (1964), Concave Programming under Linear Constraints, *Soviet Mathematics* 5, 1437–1440.

Tuy, H. (1998), *Convex Analysis and Global Optimization*, Kluwer, Dordrecht.

Zwart, P. (1971), Computational Aspects of the Use of Cutting Planes in Global Optimization, *Proceedings of 1971 Annual Conference ACM*, 457–465.